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MATHEMATICAL METHODS AND MODELLING IN INSTRUMENT MAKING

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PROPERTIES OF MEDIAN UNDER DRIFT OF ONE OF GROUP OF SENSORS AT NORMAL DISTRIBUTION

It is supposed that the data arriving from each measuring instrument are distributed under the normal law. Formulas of calculation of expected value and dispersion of median when data from one of group of measuring instruments are subject to drift are received. As a median we take the value which has appeared in the middle of the sorted list of values from the odd number of measuring instruments. It was possible to take integrals and to receive analytical formulas — when using approximate formula of integral of probability (Laplace's function). On the basis of results of numerical integration correction functions are added to the received formulas. Limits of preference of median in comparison with arithmetic average are defined.

Кл. сл.: statistics, median, normal distribution, arithmetic mean, expected value, dispersion, measurig instruments, drift

INTRODUCTION

In the previous article [1] properties of a median for uniform (rectangular) distribution of the values received from each measuring instrument (sensor) in the conditions of drift of one of group of measuring instruments are considered. Uniform distribution has allowed calculating the required integrals analytically precisely. But in practice we deal with radiation to which Poisson's distribution is peculiar. It is known that it differs from the normal distribution which is more convenient for mathematical calculations a little. In this article for normal distribution formulas of a population mean and dispersion of a median in the conditions of drift of one of group of measuring instruments are received. However these formulas are approximate. Therefore for their specification the correction functions received by comparison with results of numerical integration are added.

FORMULAS OF A GENERAL VIEW

Let $p(a,x)$ — probability density function (PDF) be set. Width of area of values is characterized by the parameter a.

We will write down also cumulative distribution function (CDF):

$$P(a,X)=\int_{-\infty}^{X} p(a,x)\mathrm{d}x .$$

We will designate N — quantity of sensors.

We will be limited to option of an odd set: N=2n+1.

We will designate L — drift size towards understating. It means that PDF takes $p(a, x+L)$ form.

We will begin with consideration of an initial state, when L=0. Formulas for this case are known from [2, pp. 17–18],[3, p. 96].

Probability of obtaining value of a median in the range from X to X+dX:

$$Q(a,n,X)\mathrm{d}X = \\ = W(n)P(a,X)^n\left(1-P(a,X)\right)^n p(a,X)\mathrm{d}X. \quad (1)$$

Here we have "shifts with repetitions" [4, p. 48] which quantity is defined by multinomial coefficient:

$$W(n)=\frac{(2n+1)!}{n!^2} .$$

The formula of the K — order moments has an appearance:

$$\mathrm{M}\left(K,a,n,0\right)=\int\limits_{-\infty}^{\infty}Q\left(a,n,X\right)X^{K}\mathrm{d}X. \qquad (2)$$

In the presence of drift the formula of calculation of the K — order moments has an appearance of the sum of three items corresponding to three options of obtaining value from the drifting sensor in comparison with a median:

$$\mathrm{M}\left(K,a,n,L\right)=R_1\left(K,a,n,L\right)+ \\ +R_2\left(K,a,n,L\right)+R_3\left(K,a,n,L\right). \qquad (3)$$

1) *Value from the drifting sensor appeared a median:*

$$R_1\left(K,a,n,L\right)= \\ =\frac{(2n)!}{n!^2}\int\limits_{-\infty}^{\infty}P\left(a,x\right)^n\left(1-P\left(a,x\right)\right)^n p\left(a,x+L\right)x^K\mathrm{d}x. \qquad (4)$$

2) *Value from the drifting sensor appeared more median:*

$$R_2\left(K,a,n,L\right)=\frac{(2n)!}{n!(n-1)!}\int\limits_{-\infty}^{\infty}\Big[P\left(a,x\right)^n\times \\ \times\left(1-P\left(a,x\right)\right)^{n-1}\left(1-P\left(a,x+L\right)\right)p\left(a,x\right)x^K\Big]\mathrm{d}x. \qquad (5)$$

3) *Value from the drifting sensor appeared less median:*

$$R_3\left(K,a,n,L\right)=\frac{(2n)!}{n!(n-1)!}\int\limits_{-\infty}^{\infty}\Big[P\left(a,x\right)^{n-1}\times \\ \times\left(1-P\left(a,x\right)\right)^n P\left(a,x+L\right)p\left(a,x\right)x^K\Big]\mathrm{d}x. \qquad (6)$$

CALCULATION OF DERIVATIVES

For a start we will analyse derivatives with respect to parameter L. They will be useful further for increase of accuracy of formulas:

$$\frac{\mathrm{d}}{\mathrm{d}L}R_1\left(K,a,n,L\right)= \\ =\frac{(2n)!}{n!^2}\int\limits_{-\infty}^{\infty}P\left(a,x\right)^n\left(1-P\left(a,x\right)\right)^n\frac{\mathrm{d}}{\mathrm{d}x}p\left(a,x+L\right)x^K\mathrm{d}x.$$

Having the derivative by x coinciding with a derivative by L in integrand expression, we will execute integration "by parts". The multiplier of x^K doesn't disturb zeroing in infinity. Therefore we receive:

$$\frac{\mathrm{d}}{\mathrm{d}L}R_1\left(K,a,n,L\right)= \\ =\frac{-(2n)!}{n!^2}\int\limits_{-\infty}^{\infty}\Big[nP\left(a,x\right)^{n-1}p\left(a,x\right)\times \\ \times\left(1-P\left(a,x\right)\right)^n p\left(a,x+L\right)x^K\Big]\mathrm{d}x+ \\ +\frac{(2n)!}{n!^2}\int\limits_{-\infty}^{\infty}\Big[P\left(a,x\right)^n n\times \\ \times\left(1-P\left(a,x\right)\right)^{n-1}p\left(a,x\right)p\left(a,x+L\right)x^K\Big]\mathrm{d}x- \\ -\frac{(2n)!}{n!^2}\int\limits_{-\infty}^{\infty}P\left(a,x\right)^n\left(1-P\left(a,x\right)\right)^n p\left(a,x+L\right)Kx^{K-1}\mathrm{d}x.$$

In the received expression the first two integrals considered with opposite signs, completely coincide with derivatives of expressions (5) and (6) by L. Therefore after reductions there is only one integral:

$$\frac{\mathrm{d}}{\mathrm{d}L}\mathrm{M}\left(K,a,n,L\right)=-K\frac{(2n)!}{n!^2}\times \\ \times\int\limits_{-\infty}^{\infty}P\left(a,x\right)^n\left(1-P\left(a,x\right)\right)^n p\left(a,x+L\right)x^{K-1}\mathrm{d}x. \qquad (7)$$

In particular, at L=0 and K=1, the received formula (7) is similar to a formula (2) at K=0 which thus is identically equal to 1. Therefore, considering difference of coefficients, we receive:

$$\frac{\mathrm{d}}{\mathrm{d}L}\mathrm{M}\left(K,a,n,0\right)=\frac{-1}{2n+1}. \qquad (8)$$

It means that at small values of drift the expected value of a median coincides with an expected value of an arithmetic average:

$$\mathrm{M}_{\mathrm{MEAN}}\left(K,a,n,L\right)=\frac{-L}{2n+1}. \qquad (9)$$

On the basis of a formula (7) further we will write down also the second derivative:

$$\frac{\mathrm{d}^2}{\mathrm{d}L^2}\mathrm{M}\left(K,a,n,L\right)=-K\frac{(2n)!}{n!^2}\times \\ \times\int\limits_{-\infty}^{\infty}P\left(a,x\right)^n\left(1-P\left(a,x\right)\right)^n\frac{\mathrm{d}}{\mathrm{d}x}p\left(a,x+L\right)x^{K-1}\mathrm{d}x.$$

In particular, at K=2 for normal distribution, using (25) and comparing with (2), we receive:

$$\frac{\mathrm{d}^2}{\mathrm{d}L^2}\mathrm{M}\left(2,a,n,0\right)=\frac{2}{(2n+1)a^2}\mathrm{M}\left(2,a,n,0\right). \qquad (10)$$

INITIAL TRANSFORMATION OF FORMULAS FOR INTEGRATION

Further it will be more convenient to us to use the following designations:

$$P(a,x)=\frac{1}{2}\left(1+\Phi(a,x)\right), \quad (11)$$

$$p(a,x)=\frac{1}{2}\Phi^{\blacklozenge}(a,x). \quad (12)$$

Limit values will be useful at integration:
$\Phi(a,-\infty)=-1,\quad \Phi(a,\infty)=1.$

The formula (2) takes a form:

$$\mathrm{M}(K,a,n,0)= =\frac{(2n+1)!}{4^n n!^2}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n p(a,x)x^K\mathrm{d}x. \quad (13)$$

Formulas (4)–(6) take a form:

$$R_1(K,a,n,L)= =\frac{(2n)!}{4^n n!^2}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n p(a,x+L)x^K\mathrm{d}x, \quad (14)$$

$$R_2(K,a,n,L)=\frac{(2n)!}{4^n n!(n-1)!}\int_{-\infty}^{\infty}\Big[\left(1-\Phi^2(a,x)\right)^{n-1}\times \times\left(1+\Phi(a,x)\right)\left(1-\Phi(a,x+L)\right)p(a,x)x^K\Big]\mathrm{d}x, \quad (15)$$

$$R_3(K,a,n,L)=\frac{(2n)!}{4^n n!(n-1)!}\int_{-\infty}^{\infty}\Big[\left(1-\Phi^2(a,x)\right)^{n-1}\times \times\left(1-\Phi(a,x)\right)\left(1+\Phi(a,x+L)\right)p(a,x)x^K\Big]\mathrm{d}x. \quad (16)$$

At addition of R_2 and R_3, removing the brackets, we find possibility of reduction. Therefore we receive:

$$R_2(K,a,n,L)+R_3(K,a,n,L)= =R_4(K,a,n,L)+R_5(K,a,n,L).$$

Here are designated:

$$R_4(K,a,n,L)= =\frac{2(2n)!}{4^n n!(n-1)!}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^{n-1} p(a,x)x^K\mathrm{d}x, \quad (17)$$

$$R_5(K,a,n,L)=\frac{-2(2n)!}{4^n n!(n-1)!}\int_{-\infty}^{\infty}\Big[\left(1-\Phi^2(a,x)\right)^{n-1}\times \times\Phi(a,x)\Phi(a,x+L)p(a,x)x^K\Big]\mathrm{d}x. \quad (18)$$

We will notice that R_4 doesn't depend on L and can be expressed as

$$R_4(K,a,n)=(2n-1)R_1(K,a,n-1,0). \quad (19)$$

At K=1 oddness of integrand expression generates identity:

$$R_4(1,a,n)=0. \quad (20)$$

We will start transformation of R_5. We consider that

$$p(a,x)=\frac{1}{2}\Phi^{\blacklozenge}(a,x),$$

$$\Phi^{\blacklozenge}(a,x+L)=2p(a,x+L).$$

Integrating "in parts", we receive:

$$R_5(K,a,n,L)=\frac{(2n)!}{4^n n!(n-1)!}\times \times\Bigg\{\frac{1}{2n}\Big[\left(1-\Phi^2(a,x)\right)^n\Phi(a,x+L)x^K\Big]_{-\infty}^{\infty}- -\frac{1}{n}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n p(a,x+L)x^K\mathrm{d}x- -\frac{K}{2n}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n\Phi(a,x+L)x^{K-1}\mathrm{d}x\Bigg\}. \quad (21)$$

In this expression at substitution of limits of integration zero turns out, and the first of two integrals on an absolute value coincides with $R_1(K,a,n,L)$ also is reduced with it. Therefore we receive:

$$\mathrm{M}(K,a,n,L)=R_4(K,a,n)-R_6(K,a,n,L). \quad (22)$$

Here it is designated:

$$R_6(K,a,n,L)= =K\frac{(2n)!}{2^{2n+1}n!^2}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n\Phi(a,x+L)x^{K-1}\mathrm{d}x. \quad (23)$$

We will notice that

$$\frac{\mathrm{d}R_6(K,a,n,L)}{\mathrm{d}L}=KR_1(K-1,a,n,L). \quad (24)$$

FORMULAS FOR NORMAL DISTRIBUTION

In the received formulas (22)–(24) there can be any smooth distribution function.

Now we will take normal distribution:

$$p(a,x)=\frac{1}{a\sqrt{2\pi}}\exp\left(-\frac{x^2}{2a^2}\right), \tag{25}$$

$$\Phi(a,x)=\operatorname{erf}\left(\frac{x}{\sqrt{2}a}\right). \tag{26}$$

The used Laplace's function (error function) has an appearance:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,\mathrm{d}t.$$

The approximate formula is known [5]:

$$\operatorname{erf}(x)\approx\sqrt{1-\exp\left(-x^2\frac{\frac{4}{\pi}+\gamma x^2}{1+\gamma x^2}\right)}. \tag{27}$$

Here it is designated:

$$\gamma=\frac{8(3-\pi)}{3\pi(\pi-4)}.$$

So bulky formula won't help to take us integrals analytically. Therefore we will use less exact, but quite acceptable option $\gamma=0$.

By analogy as in [3, pp. 103–104] wrote down an approximate PDF of a median, here we will receive approximate formulas of expected value and dispersion of a median in the conditions of drift of one of group of sensors.

The formula (26) taking into account (27) takes a form:

$$\Phi(a,x)\simeq\sqrt{1-\exp\left(-\frac{2x^2}{\pi a^2}\right)}.$$

Thus there is an opportunity to write down very convenient formula:

$$1-\Phi^2(a,x)=\exp\left(-\frac{2x^2}{\pi a^2}\right). \tag{28}$$

The formula (14) taking into account (25) and (28) takes a form:

$$R_1(K,a,n,L)=\frac{(2n)!}{4^n n!^2 a\sqrt{2\pi}}\exp\left(-\frac{L^2}{2a^2}\right)\times$$
$$\times\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{a^2}\left(\frac{2n}{\pi}+\frac{1}{2}\right)-\frac{xL}{a^2}\right)x^K\mathrm{d}x. \tag{29}$$

Before integrating, we will execute a regrouping of multipliers:

$$ux^2+vx=u\left(x+\frac{v}{2u}\right)^2-\frac{v^2}{4u}.$$

For our case we will designate:

$$u=\frac{4n+\pi}{2\pi a^2},\quad v=\frac{L}{a^2}. \tag{30}$$

Then

$$\frac{v}{2u}=\frac{\pi L}{4n+\pi}, \tag{31}$$

$$\frac{v^2}{4u}=\frac{\pi L^2}{2a^2(4n+\pi)},\quad \frac{v^2}{4u^2}=\frac{\pi^2L^2}{(4n+\pi)^2}. \tag{32}$$

We will use also properties of function of probability of normal distribution:

$$\int_0^x\exp\left(-\frac{t^2}{2a^2}\right)\mathrm{d}t=a\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2}a}\right), \tag{33}$$

$$\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2a^2}\right)\mathrm{d}x=a\sqrt{2\pi}, \tag{34}$$

$$\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2a^2}\right)x^2\mathrm{d}x=a^3\sqrt{2\pi}. \tag{35}$$

Carrying out replacement of a variable x by shift at a size (31), using (30), (32) and (34), we will transform a formula (29) at K=0 in the following look:

$$R_1(0,a,n,L)=\frac{(2n)!}{4^n n!^2}\sqrt{\frac{\pi}{4n+\pi}}\exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right).$$

On a formula (19) we will at once write down:

$$R_4(0,a,n)=(2n-1)R_1(0,a,n-1,0)=$$
$$=\frac{(2n-1)!}{4^{n-1}(n-1)!^2}\sqrt{\frac{\pi}{4(n-1)+\pi}}.$$

On the basis of a formula (24) at K=1 we carry out integration on a formula (33) :

$$R_6(1,a,n,L)=\frac{(2n)!}{2^{2n+1}n!^2}\frac{\pi a}{\sqrt{2n}}\operatorname{erf}\left(\frac{L}{a}\sqrt{\frac{2n}{4n+\pi}}\right)+c_1.$$

For a formula (29) at K=1, carrying out replacement of a variable by shift at a size (31), we find out that shift gets a multiplier role:

$$R_1(1,a,n,L)=\frac{-\pi L}{4n+\pi}R_1(0,a,n,L)=$$

$$= -\frac{(2n)!L}{4^n n!^2}\left(\frac{\pi}{4n+\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right).$$

On a formula (19) we will at once write down:

$$R_4(1,a,n) = (2n-1)R_1(1,a,n-1,0) = 0. \qquad (36)$$

On the basis of a formula (24) at K=2 we carry out integration:

$$R_6(2,a,n,L) = \pi a^2 \frac{(2n-1)!}{4^n n!^2} \times \\ \times \sqrt{\frac{\pi}{4n+\pi}}\left(\exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right)-1\right)+c_2. \qquad (37)$$

For a formula (29) at K=2, carrying out replacement of a variable by shift at a size (31), we find two nonzero members found on formulas (34) and (35), thus the square of shift gets a multiplier role:

$$R_1(2,a,n,L) = \frac{(2n)!}{4^n n!^2 a\sqrt{2\pi}} \times \\ \times \exp\left(-\frac{L^2}{2a^2}+\frac{\pi L^2}{2a^2(4n+\pi)}\right)\sqrt{2\pi}a^3\left(\frac{\pi}{4n+\pi}\right)^{\frac{3}{2}} + \\ + \frac{(2n)!}{4^n n!^2}\sqrt{\frac{\pi}{4n+\pi}}\exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right)\frac{\pi^2 L^2}{(4n+\pi)^2}.$$

This items it is possible to unite:

$$R_1(2,a,n,L) = \frac{(2n)!}{4^n n!^2}\left(\frac{\pi}{4n+\pi}\right)^{\frac{3}{2}} \times \\ \times \exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right)\left(a^2 + L^2\frac{\pi}{4n+\pi}\right).$$

However the received expression is only required to us at L=0:

$$R_4(2,a,n) = (2n-1)R_1(2,a,n-1,0) = \\ = a^2\frac{(2n-1)!}{4^{n-1}(n-1)!^2}\left(\frac{\pi}{4(n-1)+\pi}\right)^{\frac{3}{2}}.$$

We will start calculation of the constants which appeared at integration.

We will determine the constant c_1 on the basis of expected value by a formula (22) at K=1, L=0:

$$\mathrm{M}(1,a,n,0) = 0 = R_4(1,a,n) - R_6(1,a,n,0).$$

Considering (36), we receive c_1=0.

We will determine the constant c_2 on the basis of dispersion by a formula (22) at K=2, L=0.

We will notice that $R_6(2,a,n,0)=c_2$, therefore we receive the equation:

$$\mathrm{M}(2,a,n,0) = R_4(2,a,n) - c_2.$$

It means that it is possible to take c_2=0, but thus it is necessary instead of $R_4(2,a,n)$ to take $\mathrm{M}(2,a,n,0)$, calculated on the basis of a formula (2).

Of course, the formula (3) at L=0 completely coincides with a formula (2), however when using approximate expressions of probability coincidence of these formulas inevitably turns out also approximate.

Substituting (25) and (28) in (1) and (2), applying (35), we receive:

$$\mathrm{M}(2,a,n,0) = \frac{(2n+1)!}{a\sqrt{2\pi}4^n n!^2} \times \\ \times \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2a^2}\left(1+\frac{4n}{\pi}\right)\right)x^2 \mathrm{d}x = a^2\frac{(2n+1)!}{4^n n!^2}\left(\frac{\pi}{4n+\pi}\right)^{\frac{3}{2}}.$$

To full set of events there corresponds the identity $M(0,a,n,0)$=1. However when using approximate expressions of probability it is equality it is observed also approximately. Substituting (25) and (28) in (1) and (2), applying (35), we receive:

$$\mathrm{M}(0,a,n,0) = \\ = \frac{(2n+1)!}{a\sqrt{2\pi}4^n n!^2}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2a^2}\left(1+\frac{4n}{\pi}\right)\right)\mathrm{d}x = \\ = \frac{(2n+1)!}{4^n n!^2}\sqrt{\frac{\pi}{4n+\pi}}. \qquad (38)$$

It is undoubted that each of the integrals calculated here approximately, has the mistake close to an error of calculation of integral $M(0,a,n,0)$. Therefore for increase of accuracy of formulas the received expression (38) can be used as correction divider. In [7] use of this coefficient is called as normalization. Thus expressions more convenient turn out (without factorials).

We will write down only three demanded expressions:

$$R_6(1,a,n,L) = \frac{\sqrt{\pi}a}{2(2n+1)}\sqrt{\frac{4n+\pi}{2n}}\,\mathrm{erf}\left(\frac{L}{a}\sqrt{\frac{2n}{4n+\pi}}\right), \qquad (39)$$

$$R_6(2,a,n,L) = \frac{\pi a^2}{2n(2n+1)}\left(\exp\left(-\frac{2nL^2}{a^2(4n+\pi)}\right)-1\right), \qquad (40)$$

$$\mathrm{M}(2,a,n,0)=\frac{\pi a^2}{4n+\pi}. \qquad (41)$$

CALCULATION OF EXPECTED VALUE

According to a formula (22) the expression (39) taken with the sign "minus" defines an expected value of a median $M(1,a,n,L)$.

In particular, at small values of L, decomposing (39) in a row on a known formula [6, p. 119], we find out that the first member of a row coincides with (9).

We will consider also an asymptotic of expected value.

$$\mathrm{M}(1,a,n,\infty)=\frac{-\sqrt{\pi}a}{2(2n+1)}\sqrt{\frac{4n+\pi}{2n}}. \qquad (42)$$

It is obvious that at $L\to\infty$ we have the right completely to ignore the drifting sensor. Therefore pertinently to compare the asymptotic size (42) and an expected value of central (with index n-1 or n) order statistic on a set $2n$ sensors. For this purpose we will use a formula from [3, p. 96], we will write down a K-order moment formula in the following look:

$$\mathrm{M}_{2n}(K,a,n)=\frac{(2n)!}{n!(n-1)!}\times$$
$$\times\int_{-\infty}^{\infty}P(a,x)^{n-1}(1-P(a,x))^n p(a,x)x^K\mathrm{d}x. \qquad (43)$$

Taking into account a formula (11), we have:

$$\mathrm{M}_{2n}(K,a,n)=\frac{(2n)!}{2^{2n-1}n!(n-1)!}\times$$
$$\times\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^{n-1}\left(1-\Phi(a,x)\right)p(a,x)x^K\mathrm{d}x. \qquad (44)$$

Multiplier $(1-\Phi(a, x))$ gives the chance to choose from it only one item (depending on K), at which integrand expression will be even.

In particular, comparing (13) and (44), it is easy to be convinced that the identity is fair:

$$\mathrm{M}_{2n}(2,a,n)=\mathrm{M}(2,a,n-1,0). \qquad (45)$$

In particular, we have dispersion of normal distribution:

$$\mathrm{M}_{2n}(2,a,1)=\mathrm{M}(2,a,0,0)=a^2. \qquad (46)$$

Considering (12), for an expected value (at K=1) we receive the expression integrated in parts:

$$\mathrm{M}_{2n}(1,a,n)=\frac{-(2n)!}{2^{2n}n!(n-1)!}\times$$
$$\times\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^{n-1}\Phi(a,x)\Phi^{\blacklozenge}(a,x)x\mathrm{d}x=$$
$$=\frac{-(2n)!}{2^{2n+1}n!^2}\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^n\mathrm{d}x.$$

Using an approximate formula (28) and an exact formula (34), we receive:

$$\mathrm{M}_{2n}(1,a,n)=\frac{-(2n)!\pi a}{2^{2n+1}n!^2\sqrt{2n}}. \qquad (47)$$

On the basis of (44) we will write down also the expression corresponding to a full probability (at K=0):

$$\mathrm{M}_{2n}(0,a,n)=\frac{(2n)!}{2^{2n-1}n!(n-1)!a\sqrt{2\pi}}\times$$
$$\times\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^{n-1}\exp\left(-\frac{x^2}{2a^2}\right)\mathrm{d}x.$$

Using an approximate formula (28) and an exact formula (34), we receive:

$$\mathrm{M}_{2n}(0,a,n)=\frac{(2n)!}{2^{2n-1}n!(n-1)!}\sqrt{\frac{\pi}{4(n-1)+\pi}}. \qquad (48)$$

We will transform formula (47) taking into account normalization, i.e. carrying out division on (48):

$$\mathrm{M}_{2n}(1,a,n)=-\frac{\sqrt{\pi}a}{4n}\sqrt{\frac{4(n-1)+\pi}{2n}}. \qquad (49)$$

The difference between approximate expressions (42) and (49) can be considered an estimation of their accuracy. And to find the answer to a question of what of them is more preferable, it is expedient to execute numerical integration of a formula (43) at K=1 with use (25) and (26).

Meanwhile in the presence of results of numerical integration (the right column of Table 2) need of use of formulas (42) and (49) disappears. In view of their look, it is pertinent instead of a formula (39) now to write down a similar formula with the correction function $\varphi(n)$ providing coincidence to asymptotic values which are received by numerical integration. Thus we observe the requirement: at small values of L formula has to coincide with (9).

$$R_6(1,a,n,L)=$$

$$= \frac{\sqrt{\pi} a}{4n} \sqrt{\frac{4n-\varphi(n)}{2n}} \operatorname{erf}\left(\frac{2nL}{(2n+1)a} \sqrt{\frac{2n}{4n-\varphi(n)}} \right). \quad (50)$$

Values of correction function $\varphi(n)$ are presented in table 1, and the set of values of the function $R_6(1,a,n,L)$, calculated on a formula (50) is presented in table 2. For comparison under each value the result of numerical integration is specified.

In the lower line of table 2 for each value of L the result of calculation of full probability integral which has to be equal to 1 is specified. In process of increase in L there is an understating of result of numerical integration as in numerical experiment "tail" of distribution of the drifting sensor gradually left for integration interval borders. Other results of numerical integration presented in the corresponding columns are underestimated in the same measure. Nevertheless, from table 2 it is visible that the formula (50) is quite satisfactory.

Table 1. Correction function for formula of expected value of median

Function	Number n of sensors						
	3	5	7	9	11	13	15
φ	0.75772	0.81160	0.82804	0.83729	0.84038	0.84398	0.84578

Table 2. Expected value of a median depending on drift of one sensor for the normal law of distribution

Number n of sensors	L/a						
	0.5	1.0	1.5	2.0	2.5	3.0	1000.0
3	0.16294, 0.16291	0.30522, 0.30508	0.41373, 0.41332	0.48598, 0.48490	0.52800, 0.52451	0.54934, 0.53759	0.56419
5	0.09711, 0.09711	0.17859, 0.17856	0.23596, 0.23586	0.26985, 0.26946	0.28665, 0.28500	0.29364, 0.28761	0.29701
7	0.06915, 0.06915	0.12610, 0.12609	0.16473, 0.16468	0.18631, 0.18606	0.19624, 0.19514	0.20000, 0.19592	0.20155
9	0.05368, 0.05368	0.09743, 0.09742	0.12646, 0.12643	0.14217, 0.14199	0.14909, 0.14826	0.15157, 0.14849	0.15251
11	0.04387, 0.04387	0.07937, 0.07936	0.10261, 0.10258	0.11491, 0.11477	0.12019, 0.11952	0.12202, 0.11954	0.12267
13	0.03709, 0.03709	0.06696, 0.06695	0.08631, 0.08629	0.09642, 0.09630	0.10066, 0.10011	0.10210, 0.10002	0.10259
15	0.03213, 0.03213	0.05790, 0.05790	0.07448, 0.07447	0.08304, 0.08294	0.08659, 0.08611	0.08777, 0.08599	0.08816
Theoretical value P	The calculated value of a total probability P						
$P = 1$	0.99999	0.99997	0.9998	0.9988	0.9946	0.9798	–

Table 3. Conditional limit of essential advantage of a median

Function	Number n of sensors						
	3	5	7	9	11	13	15
$\left(\frac{L}{a}\right)_{0.5}$	3.3398	2.9303	2.7839	2.7084	2.6626	2.6316	2.6094

To present the course of change of expected value of median, we will find value of drift at which the expected value of a median becomes equal to a half of expected value of an arithmetic average. In other words, we will write down the equation:

$$0.5\frac{L}{2n+1}= \\ =\frac{\sqrt{\pi}a}{4n}\sqrt{\frac{4n-\varphi(n)}{2n}}\operatorname{erf}\left(\frac{2nL}{(2n+1)a}\sqrt{\frac{2n}{4n-\varphi(n)}}\right). \quad (51)$$

It is convenient to apply designation:

$$t=\frac{2nL}{(2n+1)a}\sqrt{\frac{2n}{4n-\varphi(n)}}. \quad (52)$$

The equation (51) takes a form:

$$t=\sqrt{\pi}\operatorname{erf}(t).$$

The solution of this equation is a constant:

$$t_{0.5}=1.748709.$$

Respectively we have: $\operatorname{erf}(t_{0.5})=0.98660$.
It means that in this point the expected value is close to the asymptotic value (42).

Taking into account (52) we receive the solution of the equation (51):

$$\left(\frac{L}{a}\right)_{0.5}=t_{0.5}\left(1+\frac{1}{2n}\right)\sqrt{2-\frac{\varphi(n)}{2n}}. \quad (53)$$

Results of calculations on a formula (53) are presented in table 3. When the size of drift exceeds the specified values, the median gains essential advantage in comparison with an arithmetic average.

SPECIFICATION OF SECOND ORDER MOMENTS

The formula (44) at K=2 has an appearance:

$$\mathrm{M}_{2n}(2,a,n)=\frac{(2n)!}{2^{2n}n!(n-1)!}\times \\ \times\int_{-\infty}^{\infty}\left(1-\Phi^2(a,x)\right)^{n-1}\Phi^{\bullet}(a,x)x^2\mathrm{d}x.$$

Taking into account (25) and (28) it is possible to execute integration on a formula (35):

$$\mathrm{M}_{2n}(2,a,n)=\frac{a^2(2n)!}{2^{2n-1}n!(n-1)!}\left(\frac{\pi}{4(n-1)+\pi}\right)^{\frac{3}{2}}. \quad (54)$$

We will execute a normalization, i.e. we will divide (54) on (48). We receive:

$$\mathrm{M}_{2n}(2,a,n)=\frac{\pi a^2}{4(n-1)+\pi}. \quad (55)$$

By the way, identities (45) and (46) are fair and for the received approximate formulas (41) and (55).

Having executed numerical integration of a formula (13) or (44) at K=2, we have opportunity to add correction function $\psi(n)$ to approximate formulas (41) and (55). Thus it is convenient to make designation:

$$\omega(n)=\frac{1}{4n+\pi-\psi(n)}.$$

Formulas (41) and (55) take a form:

$$\mathrm{M}(2,a,n,0)=\pi a^2\omega(n), \quad (56)$$

$$\mathrm{M}_{2n}(2,a,n)=\pi a^2\omega(n-1). \quad (57)$$

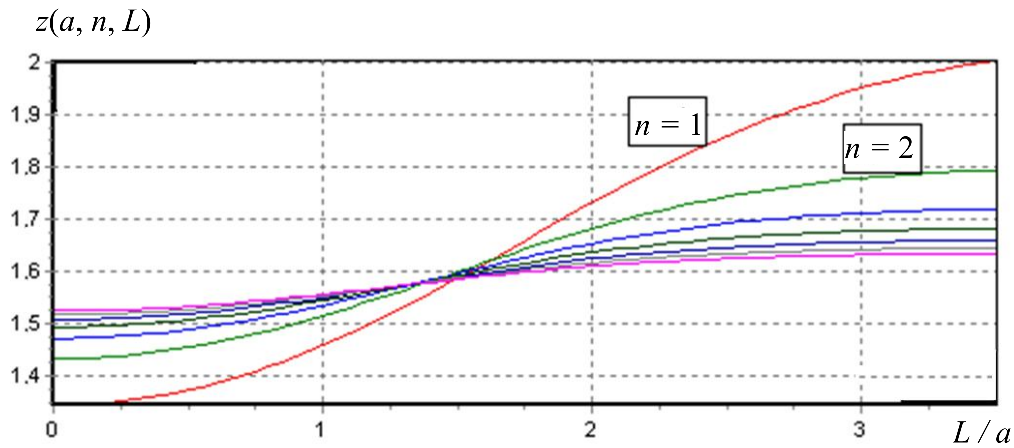
Values of correction function are presented in table 4.

We will notice that $\psi(0)=0$ according to identity (46).

Thereby we have an opportunity in a formula (40) to apply more exact coefficient which is written down on the basis of a difference between values (57) and (56). The formula (40) takes a form:

Table 4. Correction function for a formula of the moment of the second order

Function	Number n of sensors							
	1	3	5	7	9	11	13	15
ψ	0	0.13958	0.18879	0.21362	0.22773	0.23701	0.24440	0.24777



Family of curves of rated dispersion

$$R_6(2,a,n,L) = \pi a^2 \left(\omega(n-1) - \omega(n)\right) \times \left(\exp\left(-\frac{L^2\lambda(n)}{a^2}\right) - 1\right). \quad (58)$$

The demanded multiplier $\lambda(n)$ can be taken on the basis of approximate formulas (37) or (40). But it is better to use the equation (10) in which it is possible to substitute (56) and the second derivative from (58) as it defines the second derivative of the moment of the second order. As a result it is easy to receive:

$$\lambda(n) = \frac{\omega(n)}{(2n+1)\left(\omega(n-1) - \omega(n)\right)}.$$

CALCULATION OF DISPERSION

Using a known formula of calculation of dispersion [7, p. 103], considering (22), substituting (50), (56) and (58), we receive:

$$\mathrm{D}(a,n,L) = \mathrm{M}(2,a,n,L) - \mathrm{M}(1,a,n,L)^2 =$$

$$= \mathrm{M}(2,a,n,0) - R_6(2,a,n,L) - \left(R_6(1,a,n,L)\right)^2 =$$

$$= \pi a^2 \times \left[\omega(n) + \left(\omega(n-1) - \omega(n)\right)\left(1 - \exp\left(-\frac{L^2\lambda(n)}{a^2}\right)\right) - \frac{4n - \varphi(n)}{32n^3}\mathrm{erf}\left(\frac{2nL}{(2n+1)a}\sqrt{\frac{2n}{4n-\varphi(n)}}\right)^2\right].$$

For comparison we will write down dispersion of an arithmetic average

$$\mathrm{D}_{\mathrm{MEAN}}(a,n,L) = \frac{a^2}{2n+1}. \quad (59)$$

In figure the family of curves of dispersion of a median divided into dispersion of an arithmetic average is presented, by formula:

$$z(a,n,L) = \frac{\mathrm{D}(a,n,L)}{\mathrm{D}_{\mathrm{MEAN}}(a,n,L)}.$$

At n=1 the curve goes most coolly, and with increase in number of sensors the curve becomes more flat, aspiring to the horizontal line at the level $\pi/2$. Apparently, all curves have inflection points at uniform knot with coordinates approximately (1.4, 1.6).

Table 5. Limit of preference of median

Function	Number n of sensors						
	3	5	7	9	11	13	15
$\left(\frac{L}{a}\right)_{\text{PREF}}$	2.0119	2.0224	2.0861	2.1553	2.2228	2.2870	2.3480

The nature gives improvement of an expected value of a median in comparison with an arithmetic average at the price of dispersion deterioration. For the answer to a question of in what measure such price is justified, the limit of preference of use of a median is offered to be defined as the solution of the following equation containing mean square deviations, equating borders of statistical dispersion:

$$\mathrm{M}_{\mathrm{MEAN}}(1,a,n,L)+\sqrt{\mathrm{D}_{\mathrm{MEAN}}\left(a,n,L\right)}= \\ =\mathrm{M}(1,a,n,L)+\sqrt{\mathrm{D}\left(a,n,L\right)}.$$

The solution of this equation received by programming is presented in Table 5. In comparison with table 3 we see wider area of preference of a median.

CONCLUSION

The received formulas and results of calculations give opportunity to estimate median parameters in comparison with arithmetic average parameters depending on number of sensors and from drift of one sensor. Thereby possibility of a choice of the demanded number of sensors according to the set requirements for the accuracy of measurements of a random variable (dose rate) is provided.

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